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THE TRANSLATION OF A RIGID ELLIPSOIDAL INCLUSION EMBEDDED IN AN ANISOTROPIC PIEZOELECTRIC MEDIUM

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Abstract—A rigid ellipsoidal inclusion is perfectly bonded to a surrounding piezoelectric medium of infinite extent, and is translated infinitesimally by an externally imposed force. We show that the resulting exterior fields are equivalent to those induced by a layer of body force and electric charge applied over the ellipsoidal surface. Without having to solve the governing equations of equilibrium, we find directly the relation between the force and translation vectors, together with the stress, strain, rotation tensor and electric fields just outside the inclusion. Gaussian double quadratures with variable station points are employed in the numerical computations. Results are presented for two piezoelectric ceramics, GaAs and PZT-6B, to show the effect of the aspect ratio of the spheroid on the translational stiffness. This work extends the results of Walpole, L. J. (1991b) *Proc. R. Soc. London* **A434**, 571–585 to piezoelectric media.

1. INTRODUCTION

This work is a continuation of my earlier study of a rotated rigid inclusion in a piezoelectric medium (Chen, 1993a), which will be referred to as (I) in the sequel. In that work I extended Walpole's (1991a) approach and showed that the exterior fields of a rotated rigid ellipsoidal inclusion are equivalent to those induced by a layer of body force and electric charge prescribed over the ellipsoidal surface in a homogeneous piezoelectric medium. In this study I will further explore the approach in the translation of a rigid inclusion.

Piezoelectric materials, particularly piezoelectric ceramics, are an important class of engineering materials, with wide applications in actuators and sensors in "smart" materials and structures. An extensive review of the technological advantages offered by piezoelectrics is given by Smith (1989). We consider a rigid ellipsoidal inclusion embedded in a homogeneous, arbitrarily anisotropic, piezoelectric matrix and translated infinitesimally by an externally imposed force. The term "rigid" is defined in the sense that the elastic stiffness and dielectric permittivity tend to infinity so that no elastic strain or electric field is present in the inclusion. The general approach here is to let the homogeneous piezoelectric medium extend throughout the whole space. A layer of body force and electric charge is introduced over the ellipsoidal surface at a density such that the interior displacement and potential are uniform, not accompanied by any strain, rotation, or electric field. The exterior elastic and electric fields are then identical in all respects to those associated with the translated rigid inclusion. Without having to solve either the governing equations of equilibrium in the matrix or the fundamental one of a point force, we find directly the relation between the force and translation vectors, together with the stress, strain, rotation, electric field and electric displacement just outside the inclusion. The results are expressed in a closed form and evaluated numerically for arbitrary anisotropy of the medium and for arbitrary ellipticity of the inclusion. Gaussian double quadratures with a variable number of integration points are employed in the calculations. The computer routines have been checked with existing analytic solutions for transversely isotropic and isotropic solids. As an illustration, we present results for two piezoelectric ceramics, gallium arsenide and PZT-6B, to show the effect of the aspect ratio of the spheroid on the translational stiffness. Much of the analysis is relevant to the subject of interfacial discontinuities (Hill, 1983), together with the concept of body force layers originally devised by Eshelby (1957) in dealing with the ellipsoidal inclusion problem, and more recently by Walpole (1991a, b) for a rotated and translated rigid inclusion in elastic media.

Related subjects of piezoelectric inclusions and inhomogeneities have received considerable attention lately, see for example Deeg (1980), Pak (1992), Wang (1992), Benveniste (1992) and Chen (1993b). Available solutions of a translated rigid inclusion in elastic media could be found in the works by Kanwal and Sharma (1976), Selvadurai (1976, 1980, 1982) and references therein. In particular, Kanwal and Sharma employed the technique of combining and distributing suitable singularities to explore the displacement type boundary value problems. Selvadurai used Hankel integral transforms and a complex potential function approach to investigate the asymmetric displacement of a rigid elliptical disc in a transversely isotropic medium. Recent developments include a series of works by Pak and Saphores (1991, 1992), in which they examined the rotation and translation of a rigid disc in an elastic half-space by means of Hankel transforms.

Cartesian tensors will be used and their components will be written by the indicial notation, with reference to the coordinates x_1, x_2, x_3. Repeated indices indicate Einstein's summation convention with the index running from 1–3. $\mathbf{i}$ is the unit second-rank tensor δ_{ij} such that $\boldsymbol{\alpha}\boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^{-1}\boldsymbol{\alpha} = \mathbf{i}$, provided that $\boldsymbol{\alpha}$ is invertible.

2. BASIC EQUATIONS

The constitutive relation for a linear piezoelectric medium can be expressed as (Tiersten, 1969):

$$\begin{cases} \sigma_{ij} = L_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \\ D_i = e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k, \end{cases} \tag{1}$$

where σ_{ij} is the stress tensor, ε_{kl} the strain tensor, D_i the electric displacement vector, and E_i the electric field. L_{ijkl} are the elastic moduli measured in a constant electric field; κ_{ij} are the dielectric permittivities measured at constant strain; e_{ijk} are the piezoelectric constants. The material constants $\mathbf{L}$, $\mathbf{e}$, $\boldsymbol{\kappa}$ are, respectively, fourth-rank, third-rank and second-rank tensors, which satisfy the symmetry relations:

$$L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}, \quad e_{ikl} = e_{ilk}, \quad \kappa_{ij} = \kappa_{ji}, \tag{2}$$

so that L_{ijkl}, e_{ijk} and κ_{ij} admit, at most, 21, 18 and 6 independent components, respectively. If $u_i(\mathbf{x})$ is the elastic displacement vector and $\phi(\mathbf{x})$ the electric potential, the infinitesimal strain, rotation tensor and electric field are given by:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}), \quad E_i = -\phi_{,i}, \tag{3}$$

where the comma followed by an index indicates the derivative with respect to the corresponding space coordinate. In the absence of body forces and extrinsic charges the stress and electric displacement satisfy the divergence equations:

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \tag{4}$$

together with continuity of traction and normal components of electric displacement on any interface.

Similar to that of elasticity, Green's functions in piezoelectric media can be defined as (see for example, Minagawa, 1984):

$$\begin{bmatrix} L_{ijkl}\dfrac{\partial^2}{\partial x_j\,\partial x_l} & e_{kij}\dfrac{\partial^2}{\partial x_j\,\partial x_k} \\ e_{ikl}\dfrac{\partial^2}{\partial x_i\,\partial x_l} & -\kappa_{ik}\dfrac{\partial^2}{\partial x_i\,\partial x_k} \end{bmatrix} \begin{bmatrix} g^1_{kp} & g^2_k \\ g^3_p & g^4 \end{bmatrix} = \begin{bmatrix} -\delta_{ip}\delta(\mathbf{x}-\mathbf{x}') & 0 \\ 0 & -\delta(\mathbf{x}-\mathbf{x}') \end{bmatrix}, \quad (5)$$

where δ_{ij} is the Kronecker delta and $\delta(\mathbf{x}-\mathbf{x}')$ the Dirac delta function. $g^1_{ij}(\mathbf{x}-\mathbf{x}')$ and $g^3_j(\mathbf{x}-\mathbf{x}')$ are, respectively, defined to be the elastic displacement in the i direction and electric potential at $\mathbf{x}$ due to a point force applied at $\mathbf{x}'$ in the x_j direction; likewise g^2_i and g^4 are, respectively, the displacement in the i direction and electric potential at $\mathbf{x}$ due to a point charge at $\mathbf{x}'$. It is shown that, similar to that of elasticity, the Green's functions follow the elementary relations (Chen, 1993b):

$$g^1_{ij}(\mathbf{x},\mathbf{x}') = g^1_{ji}(\mathbf{x},\mathbf{x}') = g^1_{ij}(\mathbf{x}',\mathbf{x}), \quad g^4(\mathbf{x},\mathbf{x}') = g^4(\mathbf{x}',\mathbf{x}),$$
$$g^2_i(\mathbf{x},\mathbf{x}') = g^2_i(\mathbf{x}',\mathbf{x}) = g^3_i(\mathbf{x},\mathbf{x}') = g^3_i(\mathbf{x}',\mathbf{x}). \quad (6)$$

Also, it may be quoted further that the Green's functions are even homogeneous functions of the vector $(\mathbf{x}'-\mathbf{x})$ of degree minus one, namely:

$$g^1_{ij} = \hat{g}^1_{ij}(\mathbf{s})/|\mathbf{x}'-\mathbf{x}|, \quad g^2_i = \hat{g}^2_i(\mathbf{s})/|\mathbf{x}'-\mathbf{x}|, \quad g^3_i = \hat{g}^3_i(\mathbf{s})/|\mathbf{x}'-\mathbf{x}|,$$
$$g^4 = \hat{g}^4(\mathbf{s})/|\mathbf{x}'-\mathbf{x}|, \quad s_i = (x'_i - x_i)/|\mathbf{x}'-\mathbf{x}|, \quad (7)$$

where $\hat{\mathbf{g}}$ are even functions of $\mathbf{s}$ only.

3. AN ELLIPSOIDAL LAYER OF BODY FORCE AND CHARGE

An unbounded volume of a homogeneous, anisotropic piezoelectric medium is loaded by a layer of body force and charge over an internal, closed regular surface S. The origin of the Cartesian coordinate is placed conveniently at any point inside the surface S. All field variables satisfy the governing equations as described in Section 2. At the remote boundary of the medium the strain and electric field are zero. On S a layer of body force and electric charge is applied such that the discontinuity in the surface traction vector is along a constant direction A_i with a chosen variation of magnitude $x_k n_k$, and the jump in the normal component of electric displacement is a constant multiple of $x_i n_i$. That is, with no body forces or external charges elsewhere, we specify:

$$(\sigma^I_{ij} - \sigma^E_{ij})n_j = A_i x_k n_k, \quad (D^I_i - D^E_i)n_i = B x_k n_k, \quad (8)$$

at points on S, where the superscripts I and E refer to the interior and exterior parts of S, respectively; $\mathbf{n}$ is the outward unit normal to S; $\mathbf{A}$ is a constant vector and B is a scalar to be determined. Since there are no body forces or charges elsewhere, the equilibrium conditions (4) are satisfied at points inside and outside S. The resulting displacement field and electric potential are hence continuous across S and continuously differentiable elsewhere. The interfacial jumps in the displacement gradient and electric field on both sides of S can be described from the Hadamard's geometric interpretation (Hill, 1961):

$$u^I_{i,j} - u^E_{i,j} = \xi_i n_j,$$
$$\phi^I_{,i} - \phi^E_{,i} = h n_i. \quad (9)$$

On substituting (9) into (1) with reference to (8) there results:

$$c_{ik}\xi_k + d_i h = A_i x_k n_k,$$
$$d_i \xi_i - ph = B x_k n_k, \tag{10}$$

where

$$c_{ik} = L_{ijkl} n_j n_l, \quad d_j = e_{ijk} n_i n_k, \quad p = \kappa_{ij} n_i n_j. \tag{11}$$

The tensor ξ_i and scalar h are some unknowns that could be determined as:

$$\xi_i = \left(k_{ij} n_j + \frac{1}{p} k_{ij} d_j B\right) x_l n_l,$$
$$h = \frac{1}{p} d_i \left(k_{ij} A_j + \frac{1}{p} k_{ij} d_j B\right) x_l n_l - \frac{1}{p} x_l n_l B, \tag{12}$$

where

$$k_{ij} = \left(c_{ij} + \frac{1}{p} d_i d_j\right)^{-1}.$$

Substituting (12) into (9) and taking the symmetric and antisymmetric parts in (9_1), we find the jump relations:

$$\varepsilon^I_{ij} - \varepsilon^E_{ij} = \tfrac{1}{2} x_p n_p (k_{il} n_j + k_{jl} n_i) \left(A_l + \frac{1}{p} d_l B\right),$$
$$\omega^I_{ij} - \omega^E_{ij} = \tfrac{1}{2} x_p n_p (k_{il} n_j - k_{jl} n_i) \left(A_l + \frac{1}{p} d_l B\right),$$
$$E^I_i - E^E_i = -\frac{1}{p} x_p n_p n_i \left[k_{ij} d_i A_j - \left(1 - \frac{1}{p} k_{ij} d_i d_j\right) B\right], \tag{13}$$

for the strain, rotation and electric fields at point of S.

Turning to (8), the full elastic and electric fields due to the layer of body force and charge can be constructed from the reciprocal relations

$$\sigma_{ij} g^1_{ip,j} + D_i g^3_{p,i} = (L_{ijkl} g^1_{kp,l} + e_{kij} g^3_{p,k}) u_{i,j} + (e_{ikl} g^1_{kp,l} - \kappa_{ik} g^3_{p,k}) \phi_{,i},$$
$$\sigma_{ij} g^2_{i,j} + D_i g^4_{,i} = (L_{ijkl} g^2_{k,l} + e_{kij} g^4_{,k}) u_{i,j} + (e_{ikl} g^2_{k,l} - \kappa_{ik} g^4_{,k}) \phi_{,i}. \tag{14}$$

Integrating (14) over the whole of space by the divergence theorem, recalling that the displacement field and electric potential are continuous across S, and the jumps in the traction and electric charge specified in (8), we can show that:

$$u_p(\mathbf{x}) = \left[\int_S x'_j n_j g^1_{ip}(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S\right] A_i + \left[\int_S x'_j n_j g^3_p(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S\right] B,$$
$$\phi(\mathbf{x}) = \left[\int_S x'_j n_j g^2_i(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S\right] A_i + \left[\int_S x'_j n_j g^4(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S\right] B, \tag{15}$$

where the integrations are extended over the closed surface S with respect to the primed coordinates. The integrand can be simplified further by converting the integration into the interior volume of S using the divergence theorem. For example, the integral of the first term in (15) can be reduced to:

$$\int_S x_j' n_j g_{ip}^1(\mathbf{x},\mathbf{x}')\,\mathrm{d}S = \int_V (3g_{ip}^1 + x_j' g_{ip,j'}^1)\,\mathrm{d}\mathbf{x}' = \int_V (2g_{ip}^1 - x_j g_{ip,j}^1)\,\mathrm{d}\mathbf{x}', \tag{16}$$

where we have used the properties in (6) and the equivalent Euler's relation for the piezoelectric medium:

$$(x_k - x_k')g_{ij,k}^1 = -g_{ij}^1. \tag{17}$$

Thus the displacement and electric potential fields can be expressed as:

$$u_i(\mathbf{x}) = \left[\int_V (2g_{ij}^1 - x_k g_{ij,k}^1)\,\mathrm{d}\mathbf{x}'\right]A_j + \left[\int_V (2g_i^3 - x_k g_{i,k}^3)\,\mathrm{d}\mathbf{x}'\right]B,$$

$$\phi(\mathbf{x}) = \left[\int_V (2g_j^2 - x_k g_{j,k}^2)\,\mathrm{d}\mathbf{x}'\right]A_j + \left[\int_V (2g^4 - x_k g_{,k}^4)\,\mathrm{d}\mathbf{x}'\right]B. \tag{18}$$

To further simplify the expressions (18), we consider a uniform distribution of body force $\mathbf{f}$ and electric charge q acting on the whole region inside a closed regular surface S. The resulting displacement and potential fields at $\mathbf{x}$ can be represented as linear combinations of some coefficients:

$$u_i(\mathbf{x}) = G_{ij}^1(\mathbf{x})f_j + G_i^2(\mathbf{x})q, \quad \phi(\mathbf{x}) = G_i^3(\mathbf{x})f_i + G^4(\mathbf{x})q. \tag{19}$$

It was shown from the reciprocal relations (Chen, 1993a) that the functions $\mathbf{G}$ are related to the Green's tensors $\mathbf{g}$ as:

$$G_{ij}^1(\mathbf{x}) = \int_V g_{ij}^1(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}', \quad G^4(\mathbf{x}) = \int_V g^4(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}',$$

$$G_i^2(\mathbf{x}) = G_i^3(\mathbf{x}) = \int_V g_i^2(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}' = \int_V g_i^3(\mathbf{x}',\mathbf{x})\,\mathrm{d}\mathbf{x}'. \tag{20}$$

Accordingly eqn (18) can be recast as:

$$u_i(\mathbf{x}) = (2G_{ij}^1 - x_k G_{ij,k}^1)A_j + (2G_i^3 - x_k G_{ij,k}^3)B,$$

$$\phi(\mathbf{x}) = (2G_j^2 - x_k G_{j,k}^2)A_j + (2G^4 - x_k G_{,k}^4)B. \tag{21}$$

To examine the function $\mathbf{G}$ in detail, we suppose the region V is an ellipsoid expressed by $\ell_{ij}x_i'x_j' = 1$, in which ℓ_{ij} are symmetric, positive-definite constant coefficients. If $\mathbf{x}$ is located inside the region V, we can deduce the integrals (20) in a simpler form. First the volume element $\mathrm{d}\mathbf{x}'$ can be expressed as :

$$\mathrm{d}\mathbf{x}' = r^2\,\mathrm{d}r\,\mathrm{d}\omega, \tag{22}$$

where $r = |\mathbf{x}' - \mathbf{x}|$ and $\mathrm{d}\omega$ is a surface element of a unit sphere Σ centered at point $\mathbf{x}$. Next substituting $x_i' = x_i + rs_i$ in (22) and (7), upon integration with respect to r we obtain:

$$\begin{bmatrix} G_{ij}^1 & G_i^2 \\ G_j^3 & G^4 \end{bmatrix} = \frac{1}{2}\int_S r^2(\mathbf{s}) \begin{bmatrix} \hat{g}_{ij}^1(\mathbf{s}) & \hat{g}_i^2(\mathbf{s}) \\ \hat{g}_j^3(\mathbf{s}) & \hat{g}^4(\mathbf{s}) \end{bmatrix} \mathrm{d}\omega, \tag{23}$$

where $r(s)$ defines the boundary of the ellipsoid and is given by the positive root of the equation:

$$\ell_{ij}(x_i+rs_i)(x_j+rs_j) = 1, \tag{24}$$

thus:

$$r^2(s) = [2v^2+uw-2v(v^2+uw)^{1/2}]/u^2, \tag{25}$$

with

$$u = \ell_{ij}s_is_j, \quad v = \tfrac{1}{2}\ell_{ij}(s_ix_j+x_is_j), \quad w = 1-\ell_{ij}x_ix_j.$$

Since the matrix in the integrand of (23) is even with respect to $\mathbf{s}$ while the radical part of (25) is odd, their product will integrate to zero. Thus for $\mathbf{x} \in V$, functions $\mathbf{G}$ are simply quadratic functions of the coordinates:

$$\begin{bmatrix} G_{ij}^1 & G_i^2 \\ G_j^3 & G^4 \end{bmatrix} = \frac{1}{2}\left\{\begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix} - \begin{bmatrix} K_{ijkl}^1 & K_{ikl}^2 \\ K_{jkl}^3 & K_{kl}^4 \end{bmatrix} x_k x_l\right\}, \tag{26}$$

with the abbreviations:

$$\begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix} = \int_\Sigma \frac{1}{\ell_{pq}s_ps_q}\begin{bmatrix} \hat{g}_{ij}^1(\mathbf{s}) & \hat{g}_i^2(\mathbf{s}) \\ \hat{g}_j^3(\mathbf{s}) & \hat{g}^4(\mathbf{s}) \end{bmatrix} \mathrm{d}\omega,$$

$$\begin{bmatrix} K_{ijkl}^1 & K_{ikl}^2 \\ K_{jkl}^3 & K_{kl}^4 \end{bmatrix} = \int_\Sigma \left(\frac{\ell_{kl}}{\ell_{pq}s_ps_q} - \frac{2\ell_{mk}\ell_{nl}s_ms_n}{(\ell_{pq}s_ps_q)^2}\right)\begin{bmatrix} \hat{g}_{ij}^1(\mathbf{s}) & \hat{g}_i^2(\mathbf{s}) \\ \hat{g}_j^3(\mathbf{s}) & \hat{g}^4(\mathbf{s}) \end{bmatrix} \mathrm{d}\omega. \tag{27}$$

A further examination of (27) brings out the connections:

$$\begin{bmatrix} K_{ijkl}^1 & K_{ikl}^2 \\ K_{jkl}^3 & K_{kl}^4 \end{bmatrix} = \left(\ell_{kl}+2\ell_{rk}\ell_{sl}\frac{\partial}{\partial \ell_{rs}}\right)\begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix},$$

$$\begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix} = \begin{bmatrix} K_{ijkl}^1 & K_{ikl}^2 \\ K_{jkl}^3 & K_{kl}^4 \end{bmatrix} m_{kl}, \tag{28}$$

where m_{ij} is the inverse of ℓ_{ij} such that $\ell_{ik}m_{kj} = \delta_{ij}$. In deriving (28) we have employed the identity:

$$\ell_{rs}\frac{\partial}{\partial \ell_{rs}}\begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix} = -\begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix}, \tag{29}$$

which can be directly proven from the definition of (27_1).

In (I) it was shown that the coefficients $\mathbf{K}$ are the quotients of two surface integrals over S:

$$K_{ijkl}^1 = \left(\int wk_{ij}n_kn_l\,\mathrm{d}S\right)\Big/\left(\int w\,\mathrm{d}S\right),$$

$$K_{ikl}^2 = K_{ikl}^3 = \left(\int w\frac{1}{p}k_{ij}d_jn_kn_l\,\mathrm{d}S\right)\Big/\left(\int w\,\mathrm{d}S\right),$$

$$K_{ij}^4 = \left(\int w\frac{1}{p}\left[\frac{1}{p}k_{mn}d_md_n-1\right]n_in_j\,\mathrm{d}S\right)\Big/\left(\int w\,\mathrm{d}S\right), \tag{30}$$

where the weighting function $w(x)$ is the perpendicular distance from the origin to the tangent plane of the ellipsoid at each point, namely $w = x_r n_r$. Hence from (28_2) it can be readily shown that:

$$J^1_{ij} = \left(\int w^3 k_{ij}\,\mathrm{d}S\right)\bigg/\left(\int w\,\mathrm{d}S\right),\quad J^2_i = J^3_i = \left(\int w^3\frac{1}{p}k_{ij}d_j\,\mathrm{d}S\right)\bigg/\left(\int w\,\mathrm{d}S\right),$$

$$J^4 = \left(\int w^3\frac{1}{p}\left[\frac{1}{p}k_{mn}d_m d_n - 1\right]\mathrm{d}S\right)\bigg/\left(\int w\,\mathrm{d}S\right). \tag{31}$$

Note we have employed the identity of $w^2 = m_{ij}n_i n_j$ at points of S.

Returning to the displacement and potential fields in the inclusion (21), by using (26) it can be readily shown that they are reduced to the intended uniform displacement U_i and to the uniform potential Φ

$$\begin{bmatrix} U_i \\ \Phi \end{bmatrix} = \begin{bmatrix} J^1_{ij} & J^2_i \\ J^3_j & J^4 \end{bmatrix}\begin{bmatrix} A_j \\ B \end{bmatrix}. \tag{32}$$

4. A TRANSLATED RIGID ELLIPSOIDAL INCLUSION

We have shown that by prescribing a layer of body force and electric charge (8) over a closed, ellipsoidal surface inside an unbounded piezoelectric medium, the displacement and electric potential are uniform inside the region V. Accordingly, the interior displacement, electric potential and all exterior fields are equivalent in all respects to the configuration that the interior of S is filled by a rigid inclusion, which is translated infinitesimally in the direction of U_i with constant electric potential Φ. The term "rigid" is defined here in the sense that its stiffness and dielectric permittivity tend to infinity so that no elastic strain or electric field is present in the inclusion.

Since the strain and electric field (and hence the stress and electric displacement) are zero inside the ellipsoid, by referring to (8) the interfacial quantities exterior to S are:

$$\begin{bmatrix} \sigma^E_{ij}n_j \\ D^E_i n_i \end{bmatrix} = -\begin{bmatrix} A_i \\ B \end{bmatrix} x_k n_k. \tag{33}$$

The resultant force and electric charge imposed internally on the rigid inclusion by the surrounding matrix can then be evaluated by the surface integration over S:

$$\begin{bmatrix} F_i \\ Q \end{bmatrix} = \int\begin{bmatrix} \sigma^E_{ij}n_j \\ D^E_i n_i \end{bmatrix}\mathrm{d}S = -4\pi a_1 a_2 a_3\begin{bmatrix} J^1_{ij} & J^2_i \\ J^3_j & J^4 \end{bmatrix}^{-1}\begin{bmatrix} U_j \\ \Phi \end{bmatrix}, \tag{34}$$

where a_1, a_2 and a_3 are the semi-axes of the ellipsoidal surface. For convenience, we shall rewrite the relation concisely as:

$$\begin{bmatrix} F_i \\ Q \end{bmatrix} = -\begin{bmatrix} T^1_{ij} & T^2_i \\ T^3_j & T^4 \end{bmatrix}\begin{bmatrix} U_j \\ \Phi \end{bmatrix},\quad \begin{bmatrix} T^1_{ij} & T^2_i \\ T^3_j & T^4 \end{bmatrix} = 4\pi a_1 a_2 a_3\begin{bmatrix} J^1_{ij} & J^2_i \\ J^3_j & J^4 \end{bmatrix}^{-1}. \tag{35}$$

Since $J^1_{ij} = J^1_{ji}$ and $J^2_i = J^3_i$, the translation tensor $\mathbf{T}$ is also diagonally symmetric.

At points just outside S, the exterior fields of strain, rotation and electric field are derived from (13), (32) and (35) as:

$$\varepsilon_{ij}^E = -\frac{1}{8\pi a_1 a_2 a_3} x_p n_p (k_{ik} n_j + k_{jk} n_i) \left[T_{kl}^1 U_l + T_k^2 \Phi + \frac{1}{p} d_k (T_n^3 U_n + T^4 \Phi) \right],$$

$$\omega_{ij}^E = -\frac{1}{8\pi a_1 a_2 a_3} x_p n_p (k_{ik} n_j - k_{jk} n_i) \left[T_{kl}^1 U_l + T_k^2 \Phi + \frac{1}{p} d_k (T_n^3 U_n + T^4 \Phi) \right],$$

$$E_i^E = \frac{1}{p} \frac{1}{4\pi a_1 a_2 a_3} x_p n_p n_i \left[k_{ik} d_i (T_{kl}^1 U_l + T_k^2 \Phi) - \left(1 - \frac{1}{p} k_{ij} d_i d_j \right) (T_k^3 U_k + T^4 \Phi) \right]. \tag{36}$$

We have now obtained the interfacial quantities just outside the inclusion. With these boundary data one can evaluate the fields inside the matrix using boundary integral formulae [I, eqns (44) and (46)]. In particular, the problem can be treated as an infinite piezoelectric medium with an ellipsoidal cavity inside. On the surface of the cavity the stress, strain, electric field and electric displacement are all known, while at its remote boundary the strain and electric fields are vanishing. The numerical procedure generally calls for a discretization of the boundary domain and the knowledge of the Green's functions. It is beyond the present scope to discuss in detail the evaluation of the internal fields. For a detailed exposition of this particular issue, the reader is referred to standard texts of boundary element methods.

Also, it is of interest to note that the total electric enthalpy H in the whole of space $\mathscr{R}$ can be derived as:

$$\begin{aligned} H &= \frac{1}{2} \int_{\mathscr{R}} (\sigma_{ij} \varepsilon_{ij} - D_i E_i) \, \mathrm{d}v \\ &= \frac{1}{2} (J_{ij}^1 A_j + J_i^2 B) \int_S \sigma_{ij}^I n_j \, \mathrm{d}S + \frac{1}{2} (J_i^3 A_j + J^4 B) \int_S D_i^I n_i \, \mathrm{d}S \\ &\quad - \frac{1}{2} (J_{ij}^1 A_j + J_i^2 B) \int_S \sigma_{ij}^E n_j \, \mathrm{d}S - \frac{1}{2} (J_i^3 A_j + J^4 B) \int_S D_i^E n_i \, \mathrm{d}S \\ &= \frac{3}{2} V [A_i, B] \begin{bmatrix} J_{ij}^1 & J_i^2 \\ J_j^3 & J^4 \end{bmatrix} \begin{bmatrix} A_j \\ B \end{bmatrix}, \end{aligned} \tag{37}$$

where V is the volume of the inclusion.

In the case of homogeneous boundary conditions applied at infinity, the solutions can be obtained simply by superimposing a uniform field of strain and an electric field in the medium as illustrated in (I) for the rotated rigid inclusion. However, in this case the resultant couple of the inclusion will not vanish in general, and hence the inclusion acts as a translated as well as a rotated rigid one. In principle, the solutions are linear combinations of the results obtained in both works.

5. NUMERICAL RESULTS

As seen from the previous section it is obvious that the solutions rely on the evaluations of the integrals (31). Unfortunately, for arbitrary anisotropy of the medium and for arbitrary ellipticity of the inclusion it is not generally possible to obtain the results in an analytic form. So far the closed-form solutions for **J** tensors are obtained at most for a spheroid placed coaxially in an elastic, transversely isotropic medium (Walpole, 1991b).

In this work we will carry out the integrations numerically in terms of Gaussian double quadratures. Without loss in generality we suppose the principal ellipsoidal axes are aligned with the Cartesian coordinate x_1, x_2, x_3. As described in Walpole (1991b), we may introduce the following coordinate transformation:

$$x_i = a_i\zeta_i, n_i = w\zeta_i/a_i, \text{ (no sum on } i),$$
$$\zeta_1 = (1-\zeta_3^2)^{1/2}\cos\omega, \quad \zeta_2 = (1-\zeta_3^2)^{1/2}\sin\omega,$$
$$w = [(1-\zeta_3^2)(a_1^{-2}\cos^2\omega + a_2^{-2}\sin^2\omega) + a_3^{-2}\zeta_3^2]^{-1/2},$$
$$w\,\mathrm{d}S = -a_1a_2a_3\,\mathrm{d}\zeta_3\,\mathrm{d}\omega, \tag{38}$$

at points of S, so as to transform the integral J_{ij}^1 in the form:

$$J_{ij}^1 = \frac{1}{4\pi}\int_{-1}^{1}\mathrm{d}\zeta_3\int_0^{2\pi}k_{ij}(\mathbf{n})w^2\,\mathrm{d}\omega. \tag{39}$$

Other terms J_i^2, J_i^3 and J^4 can be expressed in an analogous way. Alternatively, the integrals (31) can be parameterized on the surface of a unit sphere following a coordinate transformation described as:

$$x_i = a_i^2 n_i/w, \text{ (no sum on } i),$$
$$n_1 = (1-\xi^2)^{1/2}\cos\eta, \quad n_2 = (1-\xi^2)^{1/2}\sin\eta, \quad n_3 = \xi,$$
$$w = [(1-\xi^2)(a_1^2\cos^2\eta + a_2^2\sin^2\eta) + a_3^2\xi^2]^{1/2},$$
$$w^4\,\mathrm{d}S = -a_1^2a_2^2a_3^2\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{40}$$

at points of S, and consequently arrive at the double integration:

$$J_{ij}^1 = \frac{a_1a_2a_3}{4\pi}\int_{-1}^{1}\mathrm{d}\xi\int_0^{2\pi}k_{ij}(\mathbf{n})\frac{1}{w}\,\mathrm{d}\eta, \tag{41}$$

for J_{ij}^1, and similarly for J_i^2 and others. Equations (39) and (41) can be numerically integrated using Gaussian double quadratures (see for example, Press *et al.*, 1989). In particular, eqn (41) can be approximated as:

$$J_{ij}^1 \approx \frac{a_1a_2a_3}{4\pi}\sum_{p=1}^{M}\sum_{q=1}^{N}\left[\frac{k_{ij}(\mathbf{n})}{w(\mathbf{n})}W_{pq}(\xi_p,\eta_q)\right], \tag{42}$$

where $n_i = n_i(\xi_p, \eta_q)$, M and N refer to the Gaussian points used for the integration over ξ and η, respectively, and W_{pq} are the Gaussian weights. The constants M and N can be arbitrarily selected depending on the aspect ratio of the ellipsoid, material constants and the desired accuracy. The symmetric tensor $\mathbf{J}$ in (32) may be conveniently expressed in a (4×4) matrix, and hence the $\mathbf{T}$ tensor (35), which involves the inverse of $\mathbf{J}$, could be calculated without difficulty. It should be mentioned that from dimensional considerations the $\mathbf{T}$ tensor has the unit of (length × modulus).

To check the validity of our procedures, we have compared our numerical results with existing analytic solutions for purely elastic cases. Table 1 lists the nondimensionalized

Table 1. Translational stiffness for various aspect ratios (isotropic, $\nu = 0.25$)

a_3/a_1	$T_{11}/(\pi a_1\mu)$	$T_{33}/(\pi a_1\mu)$	M	N
0.01	3.079	3.831	260	8
0.1	3.285	3.944	36	8
0.2	3.509	4.074	18	8
0.4	3.941	4.339	10	8
0.6	4.355	4.608	8	8
0.8	4.755	4.876	4	8
1.0	5.142	5.142	2	8
2.0	6.947	6.424	10	8
4.0	10.15	8.777	20	8
8.0	15.82	12.97	28	8
10.0	18.44	14.91	36	8
100.0	108.1	80.55	450	8

Table 2. Translational stiffness for various aspect ratios (eclogite, transversely isotropic)

a_3/a_1	$T_{11}/(\pi a_1 L_{44})$	$T_{33}/(\pi a_1 L_{44})$	M	N
0.01	3.033	4.236	200	8
0.1	3.233	4.352	38	8
0.2	3.450	4.484	20	8
0.4	3.867	4.755	12	8
0.6	4.266	5.029	8	8
0.8	4.651	5.302	6	8
1.0	5.025	5.572	6	8
2.0	6.761	6.876	10	8
4.0	9.851	9.275	24	8
8.0	15.30	13.56	32	8
10.0	17.81	15.54	42	8
100.0	103.9	82.42	420	8

Table 3. Elastic constants of eclogite (GPa)

L_{11}	L_{12}	L_{13}	L_{33}	L_{44}
171	60	59	208	58.5

Table 4. Material constants

		GaAs	PZT-6B
Elastic constants ($\times 10^{10}$ N m^{-2})	L_{11}	11.81	16.8
	L_{33}	11.81	16.3
	L_{44}	5.94	2.71
	L_{12}	5.32	6.0
	L_{13}	5.32	6.0
Piezoelectric constants (C m^{-2})	e_{14}	-0.16	—
	e_{15}	—	4.6
	e_{31}	—	-0.9
	e_{33}	—	7.1
Dielectric constants ($\times 10^{-10}$ F m^{-1})	κ_{11}	1.108	36
	κ_{33}	1.108	34

Table 5. Translational stiffness for various aspect ratios (GaAs)

a_3/a_1	$T^1_{11}/(\pi a_1 L_{44})$	$T^1_{33}/(\pi a_1 L_{44})$	M	N
0.01	2.534	2.844	180	24
0.1	2.676	2.939	30	24
0.2	2.832	3.050	18	24
0.4	3.138	3.284	12	24
0.6	3.437	3.525	10	24
0.8	3.727	3.768	10	24
1.0	4.011	4.011	8	22
2.0	5.342	5.191	10	22
4.0	7.725	7.369	18	24
6.0	9.893	9.374	28	24
8.0	11.92	11.26	38	26
10.0	13.87	13.06	40	26
100.0	80.21	74.73	390	26

translation tensor $\mathbf{T}_{ij}/(\pi a_1 \mu)$ vs the aspect ratio of the spheroid in an isotropic medium. The numbers M and N are the integration points necessary to achieve accuracy of four significant digits of the exact solutions (Walpole, 1991b, eqn 32). Similar results are given in Table 2 for a transversely isotropic solid—eclogite; its mechanical properties are listed in Table 3

Table 6. Translational stiffness for various aspect ratios (PZT-6B)

a_3/a_1	$T^1_{11}/(\pi a_1 L_{44})$	$T^1_{33}/(\pi a_1 L_{44})$	M	N
0.01	4.445	5.620	280	8
0.1	4.868	5.733	34	8
0.2	5.317	5.866	22	8
0.4	6.169	6.144	12	8
0.6	6.973	6.427	10	8
0.8	7.742	6.711	10	8
1.0	8.484	6.994	10	8
2.0	11.90	8.364	12	8
4.0	17.96	10.90	16	8
6.0	23.47	13.24	24	8
8.0	28.65	15.45	32	8
10.0	33.60	17.56	44	8
100.0	204.8	88.25	380	8

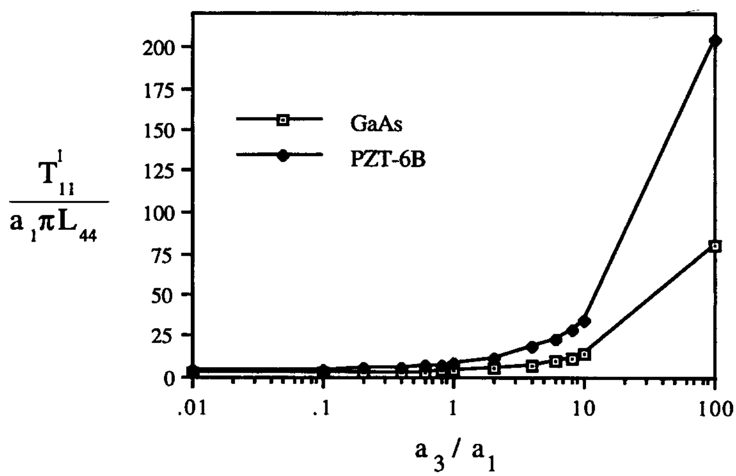

Fig. 1. Translational stiffness vs aspect ratio (about the x axis).

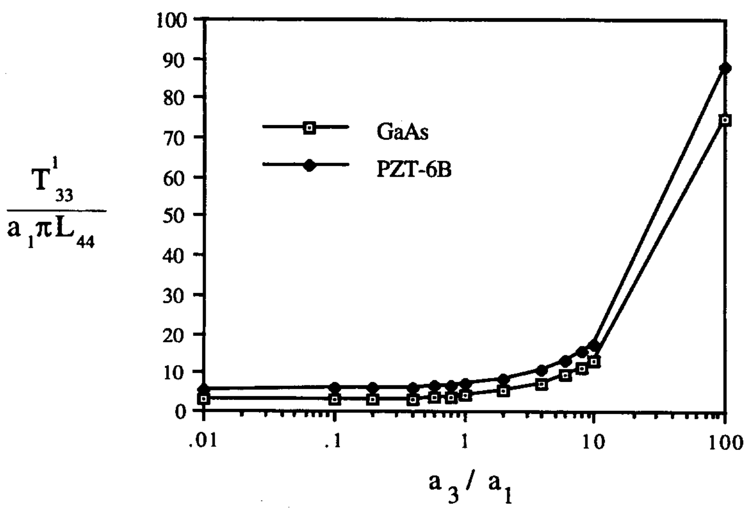

Fig. 2. Translational stiffness vs aspect ratio (about the z axis).

(Zureick and Choi, 1989). We have correctly checked with the exact analytic solutions T_{ij} given by Walpole (1991b) for a spheroidal inclusion in a transversely isotropic material. Finally, we present results in Tables 5 and 6 and in Figs 1 and 2 for two piezoelectric solids, gallium arsenide and PZT-6B. The material constants are recorded in Table 4 (Wang, 1992; Minagawa, 1992), in which the symmetry of the former corresponds to that of the cubic system of classes 23 and $\bar{4}3$, and the latter belongs to the hexagonal crystal of class 6 mm (Nye, 1957). The numbers M and N indicated are the necessary Gaussian points to achieve convergence for four significant digits. The computation time, depending on the number of station points, is within the range of seconds to few minutes on an IBM compatible 486 personal computer. This numerical approach permits efficient evaluations of the translation tensor for arbitrary anisotropy of the medium and for arbitrary aspect ratio of the ellipsoidal inclusion.

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